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Polar Convex Programming: A New Paradigm for Nonlinear Optimization

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Abstract: In this paper, our central notions of analysis are convexity and betweenness. These concepts are given from the viewpoint of conceptual spaces and are specified in relation to the coordinates of a polar system. We introduce and study a problem that seeks to optimize a polar convex objective subject to polar convex constraints, which we refer to as the polar convex programming problem. We utilize a geometric methodology to solve the two-dimensional version of this optimization problem. Our interpretation of the approach is inspired by the classical graphical method for linear optimization problems. To the best of our knowledge, this family of optimization problems has not yet been addressed or even raised by researchers.

Keywords: Polar geometry, Polar convexity, Optimization, Graphical method

1 Introduction

Operations research is a significant area of applied mathematics with a wide range of applications in business, government, engineering, and agriculture that consistently employs quantitative statistics and algorithms to carry out decision-making processes. It frequently involves the analysis of intricate real-world systems with the goal of improving (or optimizing) their performance. Given the limited resources available, operations research permits analysis of decision-making to identify ways to maximize or minimize them. The advantages of both the existing and future uses are anticipated to expand as society's citizens become more aware of the resource constraints they face and place a higher value on long-term planning and productivity gains. From everyday commercial businesses to significant choices of any kind, from engineering creation to industrial production, and from picking a career path to organizing our vacations, optimization is present everywhere. There are always some things (objectives) we are attempting to optimize in all of these activities. These objectives may include gain, cost, achievement, quality, benefit, fulfillment, and others. As a result, studies on optimization have both scientific and practical applications, and the methodology will therefore find usage in a wide range of fields.

The main component of optimization (or mathematical programming) is the formal approach to solving optimization problems. Because real-world modelers and planners tend to be pessimists, it frequently appears that real-world problems are best expressed as minimizations; however, whenever talking mathematics, maximizing problems are typically more pleasant to perform with. From the perspective of the designers (maximize benefit or minimize cost) and the analysts (maximize f or minimize $-f$), switching from one to the other is, of course, straightforward.

Among all optimization applications, linear programming [1–5] is a potent mathematical modeling tool that is frequently utilized in the oil industry, engineering design, business planning, etc. It is important to note that the term "programming" refers to planning and is unrelated to computer programming. In a broad sense, programming can be thought of as a mechanism to decide how to allocate a finite number of resources in an effort to expressibly maximize or minimize a given amount. According to Moya Navarro [6], George B.

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Dantzig and another team of affiliated academics agreed with the behest of the United States government's military authorities in 1947 and started out to research how to use mathematics and statistics to solve planning and progression problems for purely military intentions. Dantzig and his fellow workers first brought out the fundamental mathematical framework of the linear programming problem in the same year.

Finding the optimal solution to a linear goal (objective) function within linear constraints is the fundamental concept of linear programming. The variety of problems that can be handled using linear programming techniques is amazing. For instance, linear optimization has been employed to solve optimization problems with notable success in fields like management, agriculture, nutrition, planning, energy, economics, business administration, contract bidding, transportation, health care, facility location, and many others. The list is essentially infinite. Convex optimization is one of the main classes of mathematical optimization problems that comprises, but is not limited to, linear programming problems, convex quadratic programming problems (see [7] for example), and semidefinite programming problems (see [8] for example).

We concentrate in this work on the idea of convexity, which is a crucial notion in Gärdenfors' [9, 10] semantic theory: The principle that conceptions might be described as convex areas of some kind of conceptual space. The theory of conceptual space is being introduced for a wider range of reasons, including to provide a replacement for language depictions of knowledge. The conceptual spaces theory [9, 11–13] is a strategy for illustrating and describing information and is therefore not fundamentally empirical. In a conceptual space, regions represent concepts, and points represent objects. Gärdenfors [12] believed that by expressing and defining information in terms of conceptual spaces, their projectible features could be easily established. Thus, some of the more common induction problems are eschewed, and, in such a manner, better bases are provided for inductive inferences to be drawn through artificial means. He developed the theory of conceptual spaces as a specific structure for conceptually illustrating information, and he considered that the idea of conceptual space can be comprehended as an improvement over the "quality spaces" in Quine [14], the "attribute spaces" in Carnap [15], and the "logical spaces" in Stalnaker [16]. A conceptual space is built on geometrical forms based on a number of quality dimensions. A conceptual space may be expressed as a group of one or more domains. The fact that not all domains in conceptual spaces are presumed to be metric must be highlighted. In some cases, a domain consists only of a graph or an ordering with no defined distance. There is no unified scale to describe distances across the whole space; thus, the domains may be "incommensurable".

This paper's primary contribution is a new paradigm for optimization problems. The name that is attributed to it is the polar convex programming (PCP for short) problem. We propose an approach to its resolution. The focus is particularly on polarizing the graphical method from its classical applications. We present graphs at the level of polar coordinates, not Euclidean coordinates. In principle, the concept of polar betweenness (with respect to an origo) is all that is required in our study, so that the concept of a polar convex (or polarly convex) set will be defined and determined using it without the need for the metric produced by the polar coordinate system.

The present paper is organized as follows. A comprehensive exposition of all the needed facts on conceptual spaces is provided in Section 2, where we also talk about the notions of betweenness and convexity from the perspective of conceptual space, using both Cartesian and polar coordinates, which are important in our analysis. Section 3 starts with a concise discussion of linear and convex programming and moves on to their graphical method of resolution. Along the way, historical research and previous works about the topics are also taken into account. Section 4 consists of a polar convex optimization model, a complete description of the proposed method associated with the polar convex programming problem, and some examples for illustration. Finally, the conclusion part is covered in Section 5.

2 Conceptual spaces and polar convexity

As was previously indicated, in this section, we discuss convexity from the perspective of conceptual space, using both Cartesian and polar coordinates.

The set $K \subset \mathbb{R}^n$ is allegedly convex if $\alpha x + (1-\alpha)y \in K$ whenever $x, y \in K$, and $\alpha \in]0,1[$ (or equivalently, $\alpha \in [0,1]$). In geometrical terms, this signifies that the line segment between x and y is completely contained in K once both of its endpoints, x and y, are there.

The notion of "conceptual spaces" was created as a specific frame for describing knowledge at the conceptual level. Conceptual space is a high-level collection of concepts and relations, used for organizing and comparing sensory, memory, or imaginative experiences. We also have the following definition [17].

Definition 2.1. Quality dimensions are generalized distinctions which determine the kinds of domains concepts belong to, such as temperature, weight, height, width and depth.

Pitch, color, time, mass, and the three standard spatial dimensions of ordinary space (length, height, and width) are other examples of such quality dimensions. Each quality dimension is equipped with a specific metrical or topological structure. For instance, the conceptual space of Newtonian particle mechanics is, of course, based on dimensions of scientific quality rather than psychological quality. Among this theory's quality dimensions are time (similar to the line of real numbers), weight (the positive real numbers), ordinary space (3-D Euclidean), force (3-D Euclidean vector space), and mass (isomorphic to the non-negative real numbers). A particle is fully characterized in terms of Newtonian mechanics once values have been assigned to each of these dimensions. The aim is to give a realistic picture.

According to what we know, Gärdenfors [13] is the one who evolved the theory of conceptual space. The following is the conceptual space definition in Gärdenfors' vocabulary.

Definition 2.2. A conceptual space, which is a cognitive entity, consists of a number of quality dimensions. This concept is an idealized (theoretical) notion that, as a first approximation, can be thought of as the aspects or qualities of the external world that we can perceive or think about.

Any theory of representation must address the key issue of how notions ought to be modeled. Here, the major objective is to illustrate the benefits of using the conceptual level and employing conceptual spaces as a frame for the description of different situations under various circumstances. According to the conceptual approach [18], anyone can choose their own space and, in doing so, enable their favored characteristics to naturally manifest themselves there. Gärdenfors referred to his information representation method as the "conceptual form" because he thought that it best captured the key elements of concept generation. Dimensions are the fundamental structural elements of representations in conceptual space. Humans and animals can describe the qualities of objects, such as when organizing an activity, without assuming that these characteristics are expressed through an internal language or another symbolic system. In comparison to what can be realized on the symbolic level, conceptual spaces may offer a better technique for illustrating notion formation in particular and learning in general. It is considerably simpler to comprehend the workings of inductive conclusions when qualities are represented in terms of conceptual spaces. The conceptual level is appropriate for one important task assumed by symbolic representations: Providing symbolic meanings.

Another crucial aspect of representations in conceptual spaces is the need to categorize information into domains. Only a few quick observations are offered here regarding representation at the conceptual level. We now present the following observations in Remark 2.1 (see [9]):

Remark 2.1.

(i) The primary application of the theory of conceptual spaces is to function as a framework for representations. When the framework is complemented with assumptions concerning the geometrical structure of particular domains and how they are connected, one arrives at empirically testable theories.

(ii) In a conceptual space that is used as a framework for a scientific theory or for construction of an artificial cognitive system, the geometrical or topological structures of the dimensions are chosen by the scientist proposing the theory or the constructor building the system. The structures of the dimensions are tightly connected to the measurement methods employed to determine the values on the dimensions in experimental situations. Thus, the choice of dimensions in a given constructive situation will partly depend on what sensors are assumed to be used and their function.

We are so impacted by the Euclidean geometry and the Cartesian coordinate systems due to their ease of handling that we ignore that there are other ways of describing spaces. Normally, space is illustrated using the Cartesian coordinates x, y, and z, which represent width, depth, and height, respectively, and distances are measured utilizing a Euclidean metric. Nevertheless, polar coordinates are used to represent points in space by angles and distance, offering another way to depict space that may be cognitively more realistic. In the present section, we will discuss the concept of convexity from the perspective of the conceptual space, employing the Cartesian coordinates as well as the polar coordinates.

Conceptual space theory has been applied in a variety of contexts. Among these contexts, it has been utilized to define what constitutes a natural property. A natural property [12] is a convex region in some domain. A three-dimensional Cartesian space with the coordinates x, y, and z gives a formal definition for the idea of betweenness [19] as follows:

Definition 2.3. A point $b = (x_b, y_b, z_b)$ lies between a point $a = (x_a, y_a, z_a)$ and a point $c = (x_c, y_c, z_c)$ if there is some $k \in]0,1[$ such that $x_b = kx_a + (1-k)x_c$, $y_b = ky_a + (1-k)y_c$ and $z_b = kz_a + (1-k)z_c$.

With the help of the concept of "between", convexity in the conceptual space S is described in Definition 2.4 (see [9]).

Definition 2.4. A subset C of a conceptual space S is said to be convex if, for all points x and y in C, all points between x and y are also in C.

Cognitive experts believe that a polar rather than a Cartesian representation of space is more natural and has an accompanying visual interpretation as well. We are talking about this style now in order to provide a more simplistic and comprehensible approach to the graphical method subsequently. Although Definition 2.4 portrays convexity as a simple concept, Zwarts and Gärdenfors [19] reckoned that convexity is related to how spatial positions are illustrated with regard to a specific origo. They demonstrated that a specific version of convexity holds quite consistently when using polar coordinates in the domain of prepositions. Consider the three-dimensional space S established in polar coordinates, or more specifically, spherical coordinates, as seen in Figure 1.

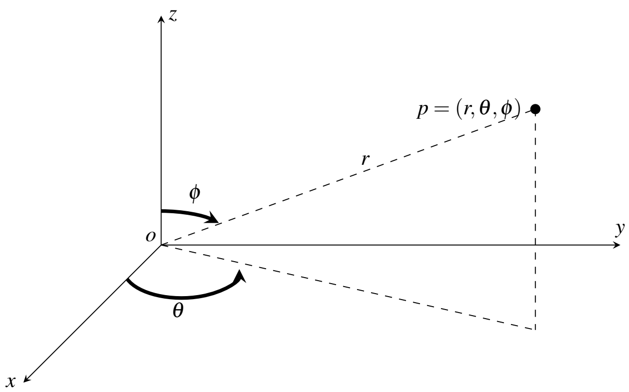

Fig. 1: Polar coordinates representation of the point $p = (r, \theta, \phi)$

The space is thought to have an origo point o. A triple (r, θ, ϕ) is utilized to represent a point p, where:

- The radius r is a non-negative real number indicating the distance between p and the origo.
- The azimuth angle θ is the angle between p and the x-axis, such that $0^\circ \leq \theta < 360^\circ$.
- The polar angle ϕ is the angle (with $0^\circ \leq \phi \leq 180^\circ$) between p and the positive z-axis (the zenith).

The angle θ is supposed to be moving counterclockwise when viewed from the positive side of the z-axis (from 'above'), in such a manner that the positive y-axis matches to $\theta = 270^\circ$ and the negative y-axis matches to $\theta = 90^\circ$. As is customary when utilizing polar coordinates, the zenith (up) and azimuth (north) serve as the fixed reference directions and are therefore two absolute settings of reference that are already incorporated into the polar coordinates.

The line segment extending rightward from the center point is called the polar axis. In terms of Cartesian coordinates, this would be known as the x-axis. The center point is referred to as the pole and correlates to a radius of $r = 0$. In the following definition, we present the polar equation of the curve.

Definition 2.5. The equation $g(\theta) = r$, which represents the dependence of the length of the radius r on the angle θ, describes a curve in the plane and is called the polar equation of the curve.

Another idea of betweenness and, thus, another concept of convexity, can be identified utilizing polar coordinates. To move forward in our discussion of polar convexity, we must first review the concept of polar betweenness (p-betweenness). We have already given a concise description of the Cartesian conception of "betweenness". Consider the following definition of polar betweenness [10], which is distinct from the one produced by the standard Euclidean metric, assuming that we are presented with a valid representation of polar coordinates as defined above.

Definition 2.6. A point $b = (r_b, \theta_b, \phi_b)$ lies between a point $a = (r_a, \theta_a, \phi_a)$ and a point $c = (r_c, \theta_c, \phi_c)$ if there is some $k \in]0, 1[$ such that

- $r_b = kr_a + (1-k)r_c$,
- $\theta_b = k\theta_a + (1-k)\theta_c$ iff $|\theta_a - \theta_c| \leq 180^\circ$, and $\theta_b = k\theta_a + (1-k)(\theta_c - 360^\circ)$ iff $|\theta_a - \theta_c| > 180^\circ$,
- $\phi_b = k\phi_a + (1-k)\phi_c$.

Although the azimuth angle can be more than 180 degrees, betweenness is determined in relation to the smallest angle. Because "p-betweenness" can only be specified to determine the shortest distance along the θ-dimension, the θ_b requirement is divided into two situations. Figure 2 illustrates that the magenta point, $(1.5, \pi/4)$, is polar between $(1.5, \pi/2)$ and $(1.5, 0)$ along a circle, which are the yellow and cyan points, respectively. Figure 3 shows that the magenta point, $(1, \pi)$, is polar between $(2, \pi)$ and $(0.5, \pi)$ along a radial line, which are the yellow and cyan points, respectively. Figure 4 shows that the magenta point, $(1.5, \pi)$, is polar between $(2, 10\pi/9)$ and $(0.5, 7\pi/9)$ along a curve, which are the yellow and cyan points, respectively.

Regarding the usual Euclidean metric, the polar coordinates add a different metric to the space. Therefore, if observed with Euclidean glasses, the lines produced by this polar betweenness relation will be curving. We make the following observation:

Remark 2.2. In the two-dimensional state, if the difference between a and b occurs only on the angle, then the points p-between them are located on a circle, as illustrated in Figure 2, where a and b are the marked points (yellow and cyan, respectively). If the only difference between a and b is their radius, the points p-between them form a radial line, as shown in Figure 3. When the radius and angle are different, then the p-between points define a curve as presented in Figure 4. Two points on opposing sides of the origo are connected by a curve around it.

Let us now examine three points that do not satisfy the property of polar betweenness. Consider Figure 5, in which three coordinate points appear within its axes. We want to prove that the magenta point, $(6, \pi)$, is not polar between $(9, 3\pi/4)$ and $(12, 5\pi/4)$, which are the yellow and cyan points, respectively. We check to see if the magenta point is polar between the other two, using the defined formulas in Definition 2.6. First, we examine whether the radius equation holds true. By replacing the values of $r_a = 9, r_b = 6,$ and $r_c = 12$ in the equation $r_b = kr_a + (1-k)r_c$, we get $k = 2$. Therefore, the definition of polar betweenness does not hold for the radius equation because k exceeds the range of its possible values. Hence, it is unnecessary to attempt to show the angular equation since it already fails for the radius. We can thus conclude that the magenta point is not polar between the yellow and cyan points in this example. Figure 5 illustrates that the magenta point, $(6, \pi)$, is non-polar between $(9, 3\pi/4)$ and $(12, 5\pi/4)$, which are the yellow and cyan points, respectively.

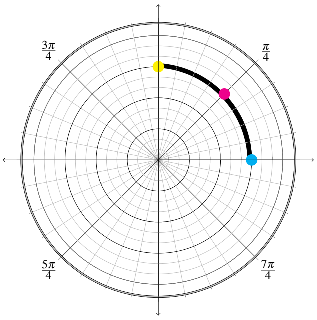

Fig. 2: Polar betweenness along a circle

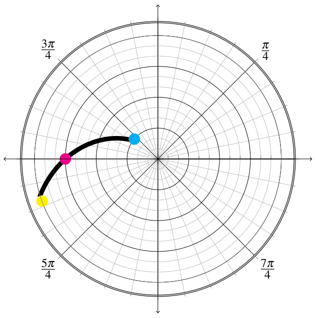

Fig. 4: Polar betweenness along a curve

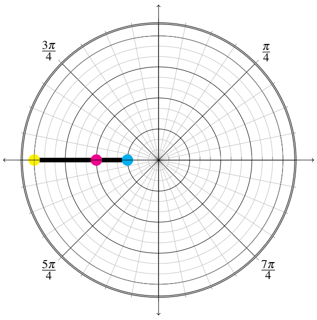

Fig. 3: Polar betweenness along a radial line

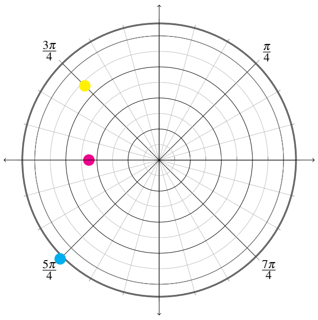

Fig. 5: Non-polar betweenness points

Consider an area R in S, which is defined as a collection of points, each of which is represented as a polar coordinates associated with a single origo o. After that, let us take a quick look at the concept of polar convexity. With the notion now invoked in Definition 2.6, we can define the property of p-convexity. We have the following formal definition [10].

Definition 2.7. A region R in S is defined to be polarly convex if and only if for all points a and b in R, any point c that is polarly between a and b is also in R.

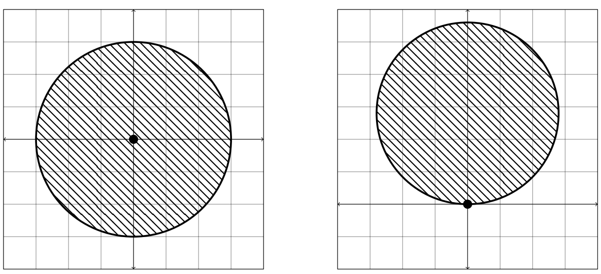

Fig. 6: A polar convex disk (left) versus a non-polar convex disk (right)

It is worthwhile noting that the convex regions formed by the Euclidean metric and those specified by the polar betweenness relation will not be the same. One idea to specify regarding Definition 2.7 is that when the Cartesian coordinate system is multiplied, translated, or rotated, Euclidean convexity is maintained. Polar convexity, on the other hand, only preserves convexity when multiplied (which changes the values of the r-axis) and rotated (the value of ϕ- and/ or θ-axes are shifting). If there is a translation, that is, if the origo is shifted, convexity could be lost. This highlights how crucial the origo is to polar coordinate systems. It should be emphasized that the whole space is the only polar convex set that permits both unrestrained rotation and unrestrained multiplication. The left side of Figure 6 illustrates an example of a polarly convex region, whereas the right side provides an illustration of a form that is not polarly convex.

Note that we can also talk about the polar convexity characteristic of the intersection between regions. If two regions are both polar convex and centered around the same origo, then their intersection is polar convex as well. A sphere is said to be p-convex only when the origin of its polar coordinates and its center coincide. The same applies to a sphere's interior, exterior, and the region between two spheres with the same origo but different radii. A number of other important three-dimensional regions are polar convex. The following remark [19] does not illustrate an exhaustive list of such regions but simply represents a small sample of eligible areas.

Remark 2.3. A full line or full plane is p-convex if it contains the origo. A half-line or line segment is p-convex if it is on the same line as the origo. A half-space or half-plane is p-convex if the origo is a point on the bounding plane or line. An infinite cone is p-convex with respect to its apex.

Conversely, there are specific regions for which polar convexity does not hold, despite the fact that they could be convex in terms of Cartesian coordinates. Some examples of figures that do not fall under the realm of polar convexity are triangles, rectangles, squares, cylinders, and cubes. An instance of a shape that is neither convex in the polar nor Cartesian sense is the double cone, as shown in [19]. We close this section with these definitions.

A curve in polar coordinates (or a polar curve) $g = g(r,\theta)$ is called polar convex if it encloses a polar convex region. For example, the circle centered at the origo, $g(r,\theta) = r$, is a polar convex curve because it encloses a disk centered at the origo, which is a polar convex set.

3 The graphical method for linear and convex optimization

Before looking at the application of the graphical method for polar convex optimization problems, we first review it for linear and convex optimization problems. The present section is devoted to a quick reminder of linear (and more generally convex) optimization problems and their methods of resolution, especially the classical graphical method, which is illustrated with two examples to facilitate the understanding of certain passages in the next section.

Linear optimization refers to the method of extracting the best outcomes from linear relationships. Problems are described in mathematical terms using optimization models. A linear (respectively, convex) optimization model consists of the problem of optimizing a linear (respectively, convex) function over linear-set (respectively, convex-set) constraints. Particularly, in linear programming problems, an optimized outcome is a maximization or minimization of the given linear representation of decision variables. This representation is known as the objective function. This function possesses certain limitations that inhibit its ability to reach values above or below specified thresholds. This event is also referred to as the constraint set.

It is known that linear and convex optimization problems have a pretty full theory, appears in a wide range of applications, and may be resolved numerically very efficiently. There are numerous methods to solve problems in linear programming. For problems with two variables, the geometric technique can be employed. In principle, the graphical approach can be used in the situation of three variables. This approach has the drawback of only solving a few specific circumstances rather than solving the entire problems of linear programming. Hundreds or thousands of decisional variables are present in many real-world problems, making them too complicated for the graphic approach to handle. Consequently, a formal strategy is required. The simplex method, developed by Danzing in 1947, was the first general approach to solving linear programming. At the boundaries of the feasible set, both graphical and simplex approaches look for maximum or minimum goal functions.

Despite the fact that practice has demonstrated that the simplex method is quite successful, in 1972, Klee and Minty [20] showed that it was not polynomial (at least with the pivot rule that is most frequently utilized, Dantzig [21]). More precisely, the unfavorable aspect of the simplex method is that, in the worst-case scenario, the total number of steps and the overall time needed to get a solution increase exponentially with the number of variables. As a result, it is claimed that the simplex approach has exponential complexity. In order to solve linear programs with polynomial complexity (that is, to find a solution in a period of time constrained by a polynomial in the number of variables), this sparked interest in creating algorithms. Khachian [22] proposed the first polynomial algorithm for resolving a linear optimization problem in 1979. But his method generated more theoretical interests than practical ones. The need to find a practical application method with polynomial complexity led Karmarkar [23] to propose a new method for linear optimization with a polynomial time in 1984. The approach looks to be more efficient than the simplex method at solving various challenging real-world planning, scheduling, and routing problems. Interior point techniques [7, 8, 24–30] are now one of the most commonly adopted for resolving linear and convex programming problems. In spite of the discovery of polynomial algorithms, the simplex method is still in use for linear programming and has undergone various modifications.

We now explain the graphical method for linear optimization. The geometric point of view bases its analysis depending on the geometry of the feasible area and makes use of concepts like convexity. It is less reliant on how the limitations are written specifically. Employing geometry makes many of the notions in linear optimization simple to grasp since they can be explained in relation to basic ideas like moving along an edge of the feasible area (especially in two-dimensional situations where a graph of the feasible region is possible). The graphical approach to tackling problems involving linear programming is briefly described below, using an example. We illustrate a maximization problem involving two inequality constraints. All values are restricted to being non-negative. Since there are only two variables, we can solve the following problem graphically.

$$\begin{aligned} \max\ & z = 2x_1 + 3x_2 \\ \text{s.t.}\ & 3x_1 + 2x_2 \leq 42, \\ & 5x_1 + 8x_2 \leq 120, \\ & x_1, x_2 \geq 0. \end{aligned} \tag{1}$$

Let us examine Problem (1). After determining the two valid sides corresponding to the constraints, the feasible region can be drawn and is shown in Figure 7. The directions of the improvement lines are displayed in Figure 8. The optimal solution is at $(48/7, 75/7)$, which is the bolded point in Figure 9. To determine the maximum value for the objective function, we input $x_1 = 48/7$ and $x_2 = 75/7$ into $z = 2x_1 + 3x_2$. As a result, the maximum value of z, given the two constraint inequalities, is $321/7$.

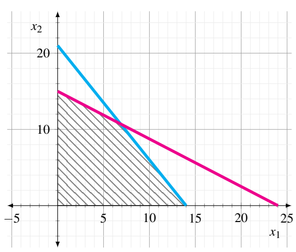

Fig. 7: The feasible region for Problem (1)

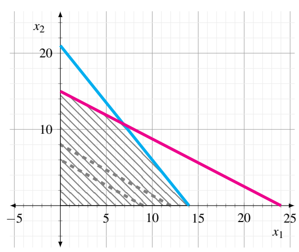

Fig. 8: The direction of improvement for Problem (1)

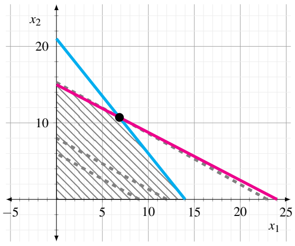

Fig. 9: The optimal solution for Problem (1)

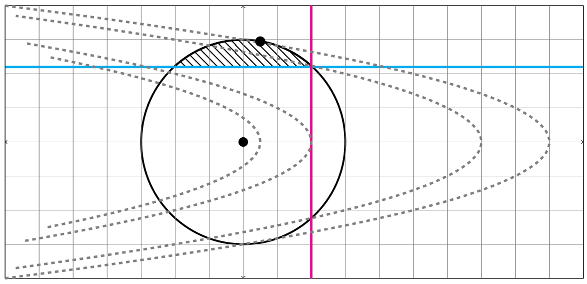

Fig. 12: The optimal solution for Problem (2)

The graphical method is not limited to solving only linear optimization problems; it is more general than that, as it is adopted to solve various types of optimization problems involving two or three variables. We started with the linear problem due to its ease on one hand and its importance on the other hand. Now we consider the following convex programming problem and solve it graphically.

$$\begin{array}{ll} \max & z = x_1 + x_2^2 \\ \text{s.t.} & x_1^2 + x_2^2 \leq 3, \\ & x_1 \quad\quad \leq 2, \\ & \quad\quad x_2 \geq \sqrt{5}. \end{array} \tag{2}$$

Figure 10 illustrates the feasible region of Problem (2), while Figure 11 represents the corresponding direction of improvement. The optimal solution is at the point $(0.5, \sqrt{8.75})$, as shown in Figure 12, and the optimal value is 9.25.

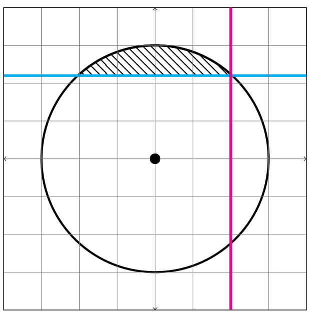

Fig. 10: The feasible region for Problem (2)

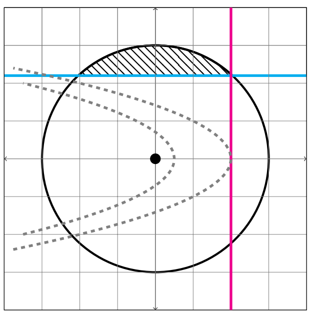

Fig. 11: The direction of improvement for Problem (2)

4 Polar convex optimization and the graphical method

It is time now to move into our key theme. In this section, the formulation of the PCP problem is given, and the proposed graphical method is illustrated using an algorithm and some examples.

The objective in a PCP problem is polarly convex curve and the feasible domain is determined by polarly convex regions. So, a PCP problem in any given situation aims to optimize the polarly convex objective under a set of polarly convex constraints. The following formal definition is what we have.

Let $g(r,\theta)$ be a polar convex curve. A polar convex optimization model is one that optimizes the objective $g(r,\theta)$ that meets a finite set of polar convex constraints. Each of the polar constraints has one of the form $h_i(r,\theta) = 0$ for $i \in \mathscr{E}$, or $h_i(r,\theta) \geq 0$ for $i \in \mathscr{I}$, where $h_i, i \in \mathscr{E} \cup \mathscr{I}$ are polar convex regions. Here, $\mathscr{E}$ and $\mathscr{I}$ are two finite sets of indices. So, a general formulation for the PCP problems is

$$\begin{array}{ll} \min_{r,\theta} & g(r,\theta) \\ \text{s.t.} & h_i(r,\theta) = 0; \ \ i \in \mathscr{E}, \\ & h_i(r,\theta) \geq 0; \ \ i \in \mathscr{I}. \end{array} \tag{3}$$

Problem (3) is the general form of PCP problems.

In Section 3, we introduced the basic context of linear optimization, especially the graphical method of its resolution, in order to make our research on PCP that will be discussed in this part easier and clearer. In this section, we lay out the initial approach to the PCP problem. We suggest a technique for graphically visualizing and analyzing p-convex problems.

The graphical method for resolving the polar convex optimization problem involves determining the minimum or maximum point(s) of the intersection on a graph in a polar coordinate system between the objective and the feasible area. The method is formulated compoundly using the themes of polar convexity and the graphical method for convex optimization encountered in the previous section.

Let us first provide a relevant example that helps with visualization before discussing the optimization method for p-convex forms. In our illustration, we have both a spherical stone and a circular pool of water. We merely need to be able to throw this stone into the water with accuracy; the dimensions of the stone need not be revealed. Initially, the stone is hurled into the exact center of the pool. It pushes the water aside for a short moment in order for the stone to create for itself a place from which it can descend to the bottom by the law of earthly gravity, and the water that propels it aside in turn thrusts the rest of the water, and this is how the circles are formed, grow, and expand in every direction (see Figure 13). Since the pool is circular and the stone is spherical, these ripples will reach the outer bounds at precisely the same time throughout the circumference. When the circles expand, their resistance gradually weakens and decreases until they end and disappear completely, at which point the water returns to a state of stillness until another stone is thrown into it. However, let us now repeat the throwing process with the same parameters observed, except for the landing spot of the stone. This time, the stone splashes down in the first quadrant of the pool. While the ripple process occurs in the same fashion as before, the outermost ripples will reach the bounds in the first quadrant before they reach those of the other quadrants. This is due to the changed "origin" point of the stone and will be essential to the method of polar convex programming.

Fig. 13: Water circles (this picture is taken from *https://webarchive.gemini.edu*)

Now that the example has been given, we can consider the idea of optimization that was previously stated. As before, this requires an objective polar convex curve along with a set of polar convex constraints that the maximization or minimization is subjected to undergo. In this section, we will deal with polar convex regions. Instead of using the Cartesian coordinate plane, we will employ the polar coordinate plane to describe and solve the polar convex optimization problems. We will see that utilizing a graphical approach makes it simple to solve PCP problems in two-dimensional polar coordinates. Because of the difficulty in visually representing or depicting polar convex problems, we limit ourselves to solving them in two dimensions using the graphical approach rather than dealing with spherical coordinates.

Algorithm 1 is stated to find the maximum or minimum value of polar convex optimization problems graphically.

Algorithm 1: The graphical method algorithm for PCP.

begin
- **Given:** Mathematical model (P) with polar coordinates (r, θ);
- **Output:** Optimal solution point (r_{opt}, θ_{opt}) and optimal value g_{opt}.
- **Test:** Is the problem (P) a polar convex program? **if** *the problem* (P) *is PCP,* **then**
  - **Procedure:**
    - (i) Graph the area that complies with all the requirements (constraints).
      - Consider inequalities as equalities, and then create a graph for each of the equalities.
      - Locate the valid side of each requirement.
      - Determine and isolate the feasible region.
    - (ii) Detect the direction of improvement by placing two polar convex curve objectives in the graph.
    - (iii) Find the coordinates of the optimal solution (optimum point) (r_{opt}, θ_{opt}).
    - (iv) Evaluate the polar convex objective curve at the acquired point (r_{opt}, θ_{opt}) to obtain the desired optimal value g_{opt}.
- **else**
  - Select an appropriate method for solving (P).
- **end if**

end

To help illustrate the optimization technique involving polar convex regions, let us look at the following examples to see how the graphical method for the PCP works. The first example that we discuss possesses a systematic approach to finding a solution that stems from the process methodology above.

Example 4.1. We consider an example involving a figure called the two-dimensional concentric circles. Concentric circles are polar convex, according to Definition 2.7, which makes them a useful case study for our method. Here, we examine an example that incorporates concentric circles, a full line, a half-space, and a circle polar convex objective curve.

$$\begin{aligned} \max\ & g = r^2 \\ \text{s.t.}\ & r \leq 3, \\ & r \geq 3/2, \\ & \sin\theta = 0, \\ & \cos\theta \leq 0, \\ & 0 \leq \theta \leq 2\pi. \end{aligned} \tag{4}$$

Now, to solve Problem (4) graphically, we need to implement the steps described in Algorithm 1 as follows:

(i) Graph the area that complies with all the constraints in a polar coordinate plane:
   – Graph constraint equations: To plot the constraints given by inequalities, we first assume that each inequality is in fact an equation. As shown in Figure 14, the brown circle represents $r = 3$, and the black circle describes $r = 3/2$, while the orange and magenta lines illustrate $\cos\theta = 0$ and $\sin\theta = 0$, respectively.
   – Determine the valid side of each constraint: Using Figure 14, we can determine which side of the brown circle we are seeking for the given inequality. Note that the first constraint is satisfied for $(1, \pi/4)$, so we are looking at the side that includes this point. We apply the same process for the three other constraints. We can directly observe that the first two constraints form concentric circles of radius 3 and 3/2, respectively. Figure 15 depicts the valid sides of constraints.
   – Isolate and identify the feasible region: The next step is completed by identifying the area of the graph that is feasible. The intersection of all valid sides is reduced to the line segment that is provided in yellow in Figure 16. It should be noted that the line segment formed by the intersection of the four polar convex regions centered on the same origo is also a polar convex line segment.

(ii) Determine the direction of improvement: Now that we have found the feasible area, we are interested in finding r_{opt} and θ_{opt} in this region that maximizes the value of the objective $g = r^2$. We strategize by making the polar convex goal equal to some arbitrary number; this arbitrary number is 1, for example, and the next step is to sketch this equation. Now that the objective has been drawn in Figure 17, we must repeat this step, but this time we will set the objective to a different value. Defining the new number depends on the objective direction, and by direction we mean whether we are focusing on a maximization or minimization objective. Because we are looking at a maximization problem, the second value that we select for g should be greater than 1. Our choice for the second value in this example is 4. We must now draw the equation $g = 4$ as well. Figure 17 shows the direction of improvement.

(iii) Find the maximum point: As shown in Figure 18, (r_{opt}, θ_{opt}) is the last point that the objective, g, hits as it moves into the feasible region toward its improvement. As a result, the combination of r_{opt} and θ_{opt} that maximizes g is $(r_{opt}, \theta_{opt}) = (3, \pi)$. So, there is a unique optimal solution to this problem.

(iv) Find the maximum value: We need to calculate the value of g for the optimal solution. We obtain the value of g_{opt} by substituting r_{opt} and θ_{opt} in $g = r^2$. The ultimate goal for Problem (4) is to find the maximum value of g within the three listed constraint inequalities and the constraint equality, which equals 9. As a result, $g_{opt} = 9$.

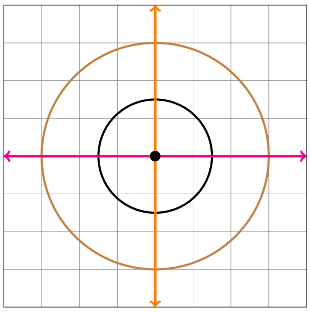

Fig. 14: The polar constraint equations in Example 4 .1

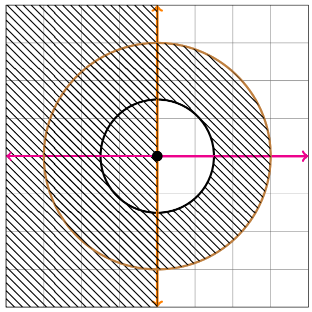

Fig. 15: The valid sides of constraints in Example 4.1

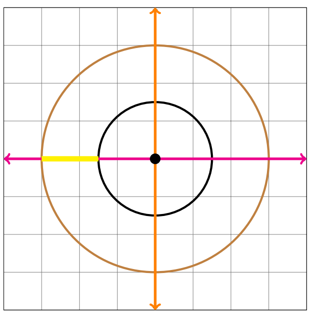

Fig. 16: The feasible region for the PCP in Example 4.1

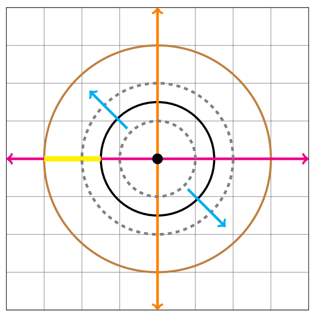

Fig. 17: The direction of improvement in Example 4.1

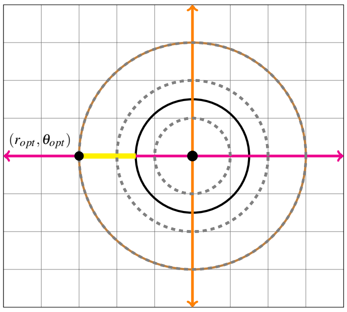

Fig. 18: The optimal solution for the PCP in Example 4.1

The PCP problem could be unbounded, feasible (with a unique solution or an infinite number of solutions), or infeasible. These four various kinds are easy to envision using the graphical method. The following are some examples.

Example 4.2. Utilize the graphical approach to address the following PCP problems.

(a) $\max g = r$
s.t. $r \geq 2$,
$\sin\theta \leq 0$,
$0 \leq \theta \leq 2\pi$.

(b) $\max g = r^2$
s.t. $r \leq 3$,
$\sin\theta \geq 0$,
$\cos\theta \leq 0$,
$0 \leq \theta \leq 2\pi$.

(c) $\max g = r$
s.t. $r \geq 3$,
$r \leq 2$,
$0 \leq \theta \leq 2\pi$.

Solution.

(a) Figure 19 illustrates the given PCP problem graphically, with the feasible region in the area shaded. The direction of improvement is described in Figure 20. Note that there is no upper limit on how far the g-circle can be augmented toward the feasible region. We can deduce from the graph shown in Figure 21 that there is no finite optimal value for g. This PCP problem is therefore unbounded.

(b) Figure 22 depicts the given PCP problem's graphic representation and the feasible region (the shaded area), while Figure 23 describes the direction of improvement. Keep in mind that the g-circle completely encircles the arc between $(3, \pi/2)$ and $(3, \pi)$. According to the graph in Figure 24, we find that the maximum value for g is 9, and that every point on the arc that is outlined in black is an optimal solution. Thus, there are an infinite number of optimal solutions to this PCP problem.

(c) Figure 25 displays the graphical representation of the given PCP problem. Note that there are no points satisfying the two constraints, i.e., there are no feasible solutions. As a result, the feasible area is empty, and the PCP problem is infeasible.

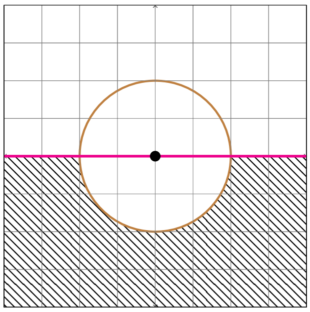

Fig. 19: The feasible region for the PCP in Example 4.2 (a)

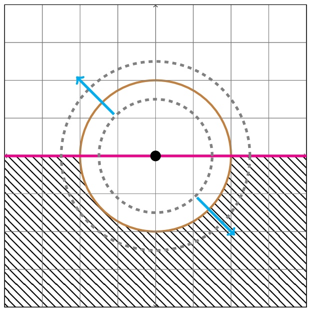

Fig. 20: The direction of improvement in Example 4.2 (a)

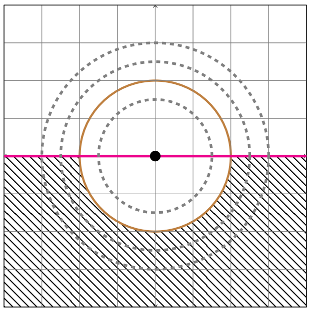

Fig. 21: The unboundedness of the PCP in Example 4.2 (a)

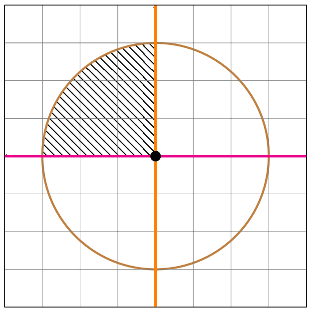

Fig. 22: The feasible region for the PCP problem in Example 4.2 (b)

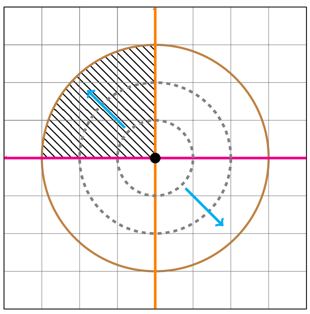

Fig. 23: The direction of improvement in Example 4.2 (b)

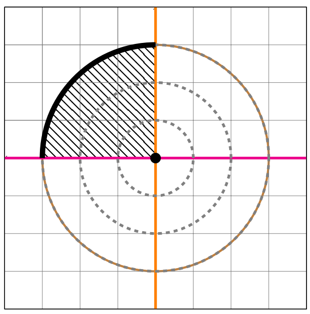

Fig. 24: The optimal solution for the PCP in Example 4.2 (b)

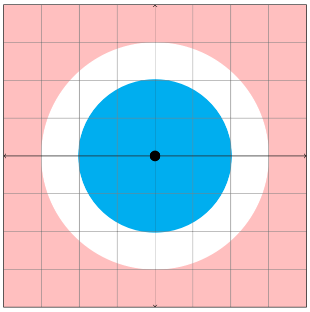

Fig. 25: The infeasibility of the PCP problem in Example 4.2 (c)

Example 4.3. Use the graphical approach to solve the following PCP problem.

$$\begin{aligned} \max\ & g = \theta \\ \text{s.t.}\ & \cos\theta + \sin\theta = 0, \\ & r \le 7/2, \\ & \cos\theta \le 0, \\ & \sin\theta \ge 0, \\ & 0 \le \theta \le 2\pi. \end{aligned} \tag{5}$$

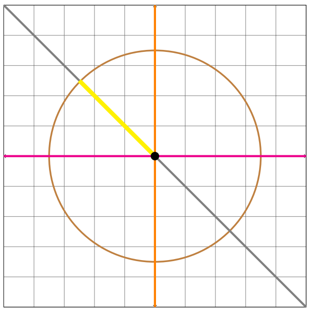

Fig. 26: The feasible region of the PCP problem in Example 4.3

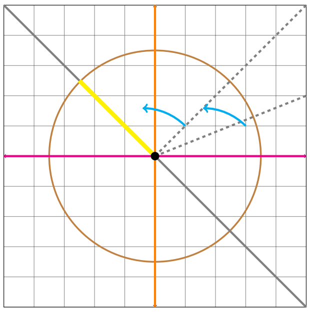

Fig. 27: The direction of improvement in Example 4.3

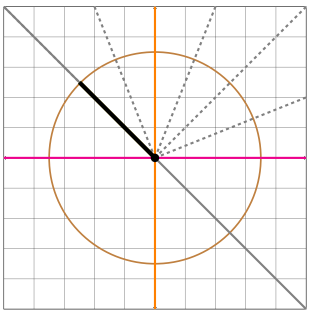

Fig. 28: The optimal solution for the PCP problem in Example 4.3

Solution. The graphical representation of Problem (5) is shown in Figure 26; the feasible area is the line segment in yellow in Figure 26; and Figure 27 presents the direction of improvement. The feasible line segment is totally covered by the g-line. The graph in Figure 28 reveals that g has a maximum value of $3\pi/4$ and that every point on the line segment that is highlighted in black is an optimal solution.

Minimization problems work in a similar manner as their maximization counterparts, except for the direction of the improvement step. Once there, we look for the direction that improves the least instead of the one that improves the most. We close this section with the following example.

Example 4.4. Consider the last example involving the other type of optimization problem, which is the minimization one.

$$\begin{aligned} \min\ & g = r \\ \text{s.t.}\ & \tan\theta + \cot\theta = 2, \\ & r \ge 1, \\ & \cos\theta \ge 0, \\ & 0 \le \theta \le 2\pi. \end{aligned} \tag{6}$$

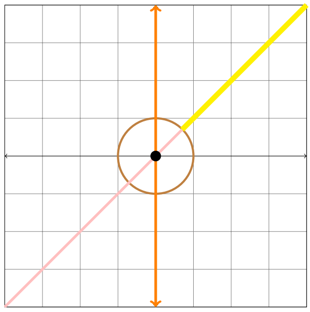

Fig. 29: The feasible region for the PCP problem in Example 4.4

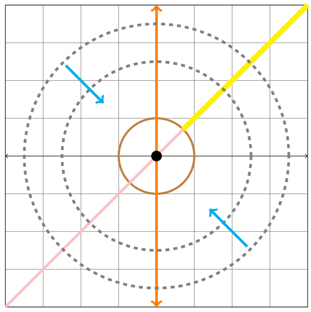

Fig. 30: The direction of improvement in Example 4.4

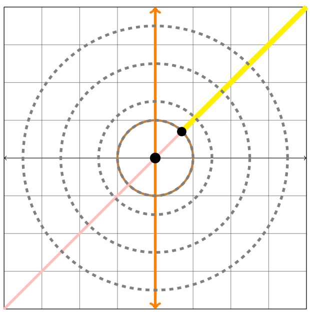

Fig. 31: The optimal solution for the PCP problem in Example 4.4

Solution. The feasible region is given by a half line pictured in yellow in Figure 29. The best solution will be found along the obtained half line. Next, we consider the direction of improvement. As done in the preceding examples, we select two arbitrary values for g. Assume $g = 3$ and $g = 5/2$, and this informs us that as g decreases, the problem moves toward minimization. To validate this claim, we will now plot two circle objectives. The two shapes are plotted on the graph shown in Figure 30. The optimal point has been bolded in Figure 31. The minimum value of g in Problem (6) is 1.

5 Conclusion

Convexity and betweenness served as our primary analytical concepts in this paper. These notions have been presented in terms of polar system coordinates and have been defined from the viewpoint of conceptual spaces. We have introduced the "polar convex optimization" problem as a new paradigm of optimization problems. In this class of problems, we optimize a polar convex objective subject to polar convex constraints. Our best understanding indicates that there are no published studies on this category of optimization problems. We have solved the two-dimensional version of this problem using a geometric approach. Our interpretation of the strategy was motivated by the traditional graphical approach to linear optimization problems. We commend ourselves and the interested researchers for looking into solving the higher-dimensional version of the PCP problem. An open question in this area of research is whether it is possible to "polarize the simplex method" for linear programming in order to be used for PCP. If this is answered affirmatively, that sounds very interesting and will be a huge motivator for further studies on this problem.

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